ON THE UNSTEADY MOTION OF GAS DRIVEN OUTWARD BY A PISTON, NEGLECTING THE COUNTER-PRESSURE

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The problem is considered of the one-dimensional unsteady motion of gas forced out by a piston moving with the speed $v = ct^m$, without consideration of the back pressure. The corresponding problem for the case $m = 0$, including the counter-pressure, was solved by Sedov [1,2] in 1945 and by Taylor [3] in 1946. Solutions of the problem of the expanding piston with speed $v = ct^m$ for the three values $m = -0.5$, -0.1, 1 were given by Krasheninnikov [4].

In the present work the problem is considered for a wide range of the number m in order to investigate the equations for various m, since it is found that depending on the ralations between ν, γ and m one obtains different pictures of the motion. If it is supposed that the motion of a mass of gas due to a strong explosion is simulated by the expansion of a cylinder or sphere with speed $v = ct^m$, for $\nu = 2$ or $\nu = 3$ we obtain the solution of the problem of a strong explosion including the forcing out of the air by the products of explosion.

1. In a quiescent gas let a plane piston begin to move at the initial moment, or the gas begin to be displaced by a cylinder or sphere with speed $v = ct^m$, where $m > -1$. We neglect the initial pressure p_1 in the undisturbed gas. The equations of one-dimensional unsteady motion of an ideal non-heat-conducting gas have the form [1]:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + (\nu - 1)\frac{\rho v}{r} = 0$$

$$\frac{\partial}{\partial t}\frac{p}{\rho^\gamma} + v\frac{\partial}{\partial r}\frac{p}{\rho^\gamma} = 0 \qquad (1.1)$$

where γ is the adiabatic exponent, and $\nu = 1, 2, 3$ respectively for plane, cylindrical or spherical symmetry.

From the characteristic parameters of the problem

$$[r] = L,\quad [t] = T,\quad [\rho] = ML^{-3},\quad [c] = LT^{-m-1}$$

it is possible to form only one dimensionless combination

$$\lambda = \frac{ct^{m+1}}{r} \tag{1.2}$$

and the problem is consequently self-similar. For the unknown functions v, ρ and p one can write the formulas [1]

$$v = \frac{r}{t} V(\lambda),\qquad \rho = \rho_1 R(\lambda),\qquad p = \rho_1 \frac{r^2}{t^2} P(\lambda) \tag{1.3}$$

If we substitute v, p and ρ from (1.3) into (1.1) and introduce in place of p the new function $z = \gamma P/R$, the system (1.1) is transformed into [1,4]

$$\frac{dz}{dV} = \frac{z[2(V-1) + \nu(\gamma-1)V](V-m-1)^2}{(V-m-1)[V(V-1)(V-m-1) - (2m/\gamma + \nu V)z]} -$$

$$- \frac{z\{(\gamma-1)V(V-1)(V-m-1) + 2z(V-m-1+m/\gamma)\}}{(V-m-1)[V(V-1)(V-m-1) - (2m/\gamma+\nu V)z]} \tag{1.4}$$

$$\frac{dR}{dV} = \frac{R}{(V-m-1)}\left\{\frac{[(V-m-1)^2 - z]\nu V}{V(V-1)(V-m-1) - (2m/\gamma+\nu V)z} - 1\right\} \tag{1.5}$$

$$\frac{d\lambda}{dV} = \lambda \frac{(V-m-1)^2 - z}{V(V-1)(V-m-1) - (2m/\gamma+\nu V)z} \tag{1.6}$$

From the conditions on the shock wave we obtain expressions for V_2 z_2 and R_2 behind the shock front (ahead of the front $V_1 = z_1 = 0$, $R_1 = 1$):

$$V_2 = \frac{2(m+1)}{\gamma+1},\qquad z_2 = \frac{2\gamma(\gamma-1)(m+1)^2}{(\gamma+1)^2},\qquad R_2 = \frac{\gamma+1}{\gamma-1} \tag{1.7}$$

We find the boundary conditions on the piston:

$$v_* = \frac{dr_*}{dt} = \frac{r_*}{t} V_* = ct^m,\quad \text{or}\quad r_* = \frac{c}{m+1} t^{m+1}$$

From (1.2) and (1.3) we obtain

$$V_* = m+1,\qquad \lambda_* = m+1 \tag{1.8}$$

(an asterisk indicates values on the piston).

Thus the problem is reduced to the integration of the system of three ordinary differential equations (1.4)-(1.6) for the unknown functions z, λ and R with boundary conditions (1.7) and (1.8).

2. For the solution of the problem it is necessary to carry out a qualitative investigation of equation (1.4) and to find from equation (1.6) the direction of increase of the parameter λ along the integral curves. The straight lines $V = m + 1$ and $z = 0$ are members of the family

of integral curves.

It turns out that in the region $0 < V < m + 1$, $z > 0$ equation (1.4) may have the following singular points:

(1) Point $O(V = 0,\ z = 0)$, a nodal point. In the neighborhood of point O the integral curves have the form

$$V + \frac{2m}{(m+1)\gamma} z = Cz^{1/2} \tag{2.1}$$

(2) Point $C(V = m + 1,\ z = 0)$, a complicated singular point. The asymptotic formula has the form

$$(V - m - 1)^{2m} = Cz^{\nu(m+1)}[\nu z - m\gamma(V - m - 1)]^{2m+\nu(\gamma-1)(m+1)} \tag{2.2}$$

(3) Point $D(V = m + 1,\ z = \infty)$. Near this point

$$z = C(m + 1 - V)^{\frac{2m}{2m+\nu\gamma(m+1)}} \tag{2.3}$$

(4) Point $E(V = -2m/\nu\gamma,\ z = \infty)$

(5) Point $G(V = 1,\ z = 0)$

(6) Point Fi is found as an intersection of the curves

$$z = \frac{\{[2(V-1) + \nu(\gamma-1)V](V - m - 1) - (\gamma-1)V(V-1)\}}{2(V - m - 1 + m/\gamma)}(V - m - 1)$$

$$z = \frac{V(V-1)(V-m-1)}{2m/\gamma + \nu V}$$

With variation of the parameter m from -1 to ∞ the character of these singularities changes, four characteristic cases being found.

First case, $m > 0$. Points O, C, G are nodes, D and F are saddle points, and point E is not singular. The integral curve of the problem of the piston [passing through the point $(V_2,\ z_2)$] enters point C, where the asymptote has the form

$$z = C(m + 1 - V)^{\frac{2m}{2m+\nu\gamma(m+1)}} \tag{2.4}$$

The solution of the problem exists for all m.

The case $m = 0$ was investigated by Sedov [1,2]. Points O, C and $A(V = 0,\ z = 1)$ are nodes, and F and D are saddle points. The straight line $V = 1$ is not included in the integral curves. Since in this case the problem is self-similar for arbitrary p_1, that is, for arbitrary z_1, the solution of the problem in the $(V,\ z)$ plane is given by an arbitrary segment of the integral curve between the points $(1,\ z)$ and $(V_2,\ z_2)$, where V_2 and z_2 are connected by the relation

$$z_2 = (1 - V_2)\left(1 + \frac{\gamma - 1}{2} V_2\right)$$

Second case, $m'' < m < 0$. Here

$$m'' = -\frac{\nu(\gamma-1)}{2+\nu(\gamma-1)} \tag{2.5}$$

The points O and D are nodes, and E a saddle point. Three integral curves pass through point C: the straight lines $V = m + 1$ and $z = 0$ and a certain dividing curve entering at the angle $-m\gamma/\nu$; consequently, this point corresponds to two saddle points. There is an odd number (always at least one) of the points Fi, of which the nearest to the point C is a node, and the further ones alternately saddle points and nodes. For

$$m = m' = -\frac{\nu}{2+\nu} \tag{2.6}$$

the field of integral curves coincides with the field for a strong explosion. The equation of the integral curve (coinciding in this case with the dividing curve entering point C) has in the (V, z) plane the form [1]

$$z = \frac{(\gamma-1)}{2} \frac{V^2[V - 2/(2+\nu)]}{[2/(2+\nu)\gamma - V]} \tag{2.7}$$

For $m = m'$ the point (V_2, z_2) lies on the dividing curve, for $m > m'$ it lies between the straight line $V = m + 1$ and the dividing curve, and for $m < m'$ it lies between the dividing curve and the z axis.

If $\gamma < 2$, the value $m = m'$ does not belong to the range of values of m considered, and the solution of the problem of the piston exists. The integral curve passes through point D.

If $\gamma > 2$, with $m < m'$, moving along the integral curve from the point (V_2, z_2) in the direction of increase of the parameter λ, we reach neither point C nor the point D, where $V = m + 1$; consequently the solution of the piston problem does not exist in this case. We devote further attention to the case $m = m'$. Point F has the coordinates

$$V = \frac{2}{2+\nu(\gamma-1)}, \qquad z = \frac{2\nu\gamma(\gamma-1)(\gamma-2)}{[2+\nu(\gamma-1)]^2[\nu+2(\gamma-1)]}$$

With $\nu = 1$ or $\nu = 2$ and also with $\nu = 3$ if $\gamma < 7$, the point (V_2, z_2) lies between points E and F; with $\nu = 3$ and $\gamma = 7$, the point (V_2, z_2) coincides with point F; and with $\nu = 3$ and $\gamma > 7$ it lies between points F and C. Consequently the solution of the piston problem exists for $\nu = 3$ and $\gamma > 7$.

For $\gamma > 7$ the solution is given by the segment of the integral curve (2.7) between the point (V_2, z_2) and point C; that is, it coincides with an appropriate solution of the problem of a strong explosion. The pressure and density are equal to zero on the surface of the piston; consequently, the piston may be replaced by an empty cavity.

For $\gamma = 7$ the solution of the problem of a strong explosion is given in the (V, z) plane by the single point $(V_2 = 0.1, z_2 = 0.21)$, whereas the solution of the piston problem is given by the segment of the integral curve (2.7) between this point and point C. At the point (V_2, z_2), $\lambda = 0$; consequently the shock wave moves away at once to infinity.

Thus, for $\gamma < 2$ the solution of the piston problem exists for all m in the range considered. With $\gamma > 2$ in the cases $\nu = 1$ or $\nu = 2$, and with $2 < \gamma < 7$ in the case $\nu = 3$, it does not exist for $m'' < m < m'$; nor does it with $\gamma > 7$ and $\nu = 3$ for $m'' < m < m'$, where m' and m'' are given by (2.5) and (2.6).

Third case, $m''' < m < m''$. Here

$$m''' = -\frac{\nu\gamma}{2+\nu\gamma} \tag{2.8}$$

points O, C and D are nodes, E a saddle point, and point F is not a singularity. We consider first the case $\gamma < 2$.

The value of the parameter $m = m'$ belongs to the range of m under consideration. For this value of m the solution of the piston problem is given by the segment of the integral curve (2.7) between the points (V_2, z_2) and C. However, in moving from the point (V_2, z_2) to C the parameter λ decreases. Such a family of motions represents for $t < 0$ the motion behind a shock front arising from a peripheral explosion. This case was considered by Grodzovskii [5].

For $m > m'$ a solution exists not only for a diverging shock wave (with the integral curve entering point D), but also for a converging one. For $m''' < m < m'$ a solution exists only for a converging wave. (The motion behind the front of a converging wave for $t < 0$ can be regarded as arising from a peripheral explosion with the motion of the products of explosion neglected).

For $\gamma > 2$ the motion involves only a converging shock wave.

Fourth case, $-1 < m < m'''$. Points O and C are nodes and D a saddle. The solution exists only with a converging shock wave. In all cases of converging shock waves the integral curve passes through the point C.

Thus, from the results of the present section we can draw the following conclusions:

(1) For $m > m'$ the solution of the piston problem exists for all ν and γ.

(2) For $\nu = 3$, $\gamma > 7$ the solution of the piston problem exists also for $m = m'$. If $\gamma \neq 7$, it coincides with the solution of the problem of the strong explosion.

(3) The solution of the problem of a peripheral explosion neglecting the motion of the products of explosion exists for $-1 < m < m''$.

(4) For $\gamma < 2$ and every m there exists a solution of the piston problem with either a diverging or converging shock wave.

(5) For $\gamma > 2$ and $m'' < m < m'$ there exists no solution of the piston problem with either a diverging or converging shock wave.

(6) Furthermore, for $m = m'$ there exists no solution or the piston problem with either a diverging or converging shock wave for $\nu = 1$ or 2 or for $\nu = 3$, $2 < \gamma < 7$.

The first of these conclusions was obtained by Grigorian [6], and also by Lees *et al* [7], who considered values of the parameter $m' < m < 0$.

We note [7,8,9] that the problem of steady flow past a plane ($\nu = 1$) or axisymmetric ($\nu = 2$) slender body at high supersonic speeds is equivalent to the problem of one-dimensional unsteady motion of gas forced out by a plane or cylindrical piston respectively, with speed $U = V \tan a$. Here the coordinate x in the stream direction must be replaced by Vt, and a is the angle between the stream direction and the tangent to the surface of the body. The solutions with converging waves give the flow past slender ducted bodies.

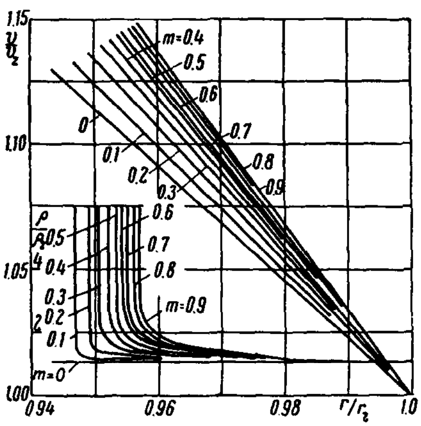

Fig. 1.

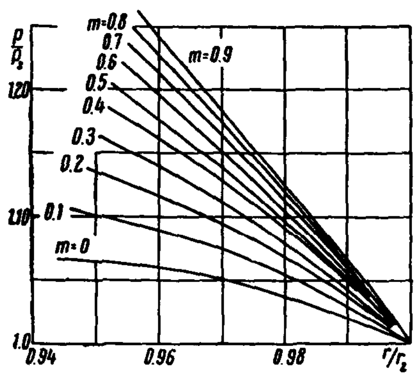

Fig. 2.

Thus for $m' < m < 9$ the piston problem considered here is equivalent to the problem of flow at high supersonic speeds past slender blunted plane profiles ($\nu = 1$) or bodies of revolution ($\nu = 2$) of the form $r \quad x^{m+1}$. Here the shock wave is similar in form to the body. In the case of converging shock waves the distance from the shock wave to the axis of symmetry is less than the corresponding distance for the body. Grodzovskii [5] has considered the case of self-similar flow past a body producing a parabolic shock wave $x = kr^2$.

3. We present some results of integrating the system of equations (1.4) - (1.6) with the boundary conditions (1.7) and (1.8) for values of m increasing by 0.1 from -0.9 to $+0.9$ with $\nu = 3$ and $\gamma = 1.4$. Fig. 1 to 8 show graphs of the functions v/v_2, ρ/ρ_2, p/p_2 (where the subscript 2 indicates the characteristics of the motion immediately behind the front of the shock wave). From relations (1.2) and (1.3) it follows that

$$\frac{v}{v_2} = \frac{\lambda_2}{\lambda}\frac{V}{V_2}, \quad \frac{\rho}{\rho_2} = \frac{R}{R_2}, \quad \frac{p}{p_2} = \left(\frac{\lambda_2}{\lambda}\right)^2 \frac{zR}{z_2 R_2}, \quad \frac{T}{T_2} = \left(\frac{\lambda_2}{\lambda}\right)^2 \frac{z}{z_2}, \quad \frac{r}{r_2} = \frac{\lambda_2}{\lambda}$$

The graphs indicate how the characteristics of the motion depend on the parameter m.

In connection with the four characteristic types of behavior of the integral curves considered in the previous section and the conclusions drawn for $\gamma < 2$, we obtain three types of graphs, namely: Fig. 1 and 2 correspond to case 1, Fig. 3-5 to case 2 and to case 3 with $m' < m < m''$, Fig. 6-8 to case 3 with $m'''< m < m'$ and to case 4. Consequently, Fig. 1-5 correspond to motions with diverging shock waves, and Fig. 6-8 to converging ones.

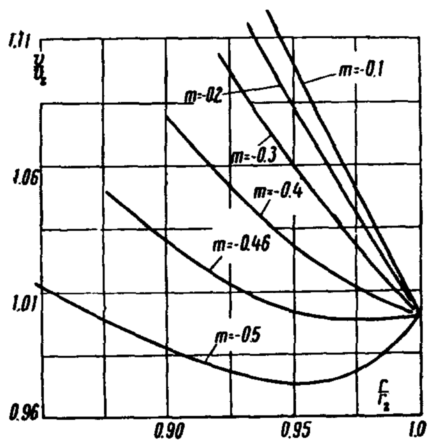

Fig. 3.

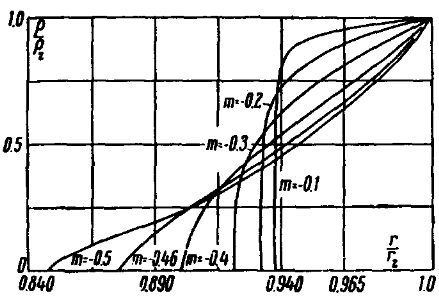

Fig. 4.

It is evident from the graphs that the speed of a particle of gas adjacent to the piston is greater than the speed of a particle of gas immediately behind the shock wave. The relative distance between the shock wave and the piston $(r_2 - r_*)/r_2$ depends only weakly on m, growing as m increases.

As is known [1], for self-similar motions the radius of the shock front r_2, its speed D, and the pressure p_2 behind the shock front are determined by the formulas

$$p_2 = \frac{2\rho_1}{\gamma+1} D^2, \quad r_2 = \frac{ct^m}{\lambda_2}, \quad D = \frac{dr_2}{dt} = \frac{(m+1)\,ct^m}{\lambda_2}$$

Hence it is clear that for $m > 0$ the speed D of the front increases with increasing t from zero to infinity. There the counter-pressure p_1 can be neglected in comparison with the pressure p_2 at the shock front only for large t. It is evident from the graphs that in this case the pressure on the piston exceeds at any time the pressure at the shock front. The density at the piston is infinite, the temperature is infinitesimally small, and the pressure is finite (Fig. 1, 2).

For $m = 0$ the shock wave and piston move with constant speed. The pressure, density and temperature on the piston are finite. In this case the motion is self-similar even including the counter-pressure, since $[c] = LT^{-1}$; $[\rho_1] = ML^{-3}$, $[p_1] = ML^{-1}T^{-2}$ and, consequently, $[c^2] = [p_1/\rho_1]$. The solution of the problem of the motion of gas forced outwards by a sphere expending with constant speed was first given by Sedov [2] in 1945.

For $m < 0$ the speed of the shock wave decreases with time from infinity to zero; consequently the counter-pressure can be neglected for times near the beginning, when the pressure at the shock front is great.

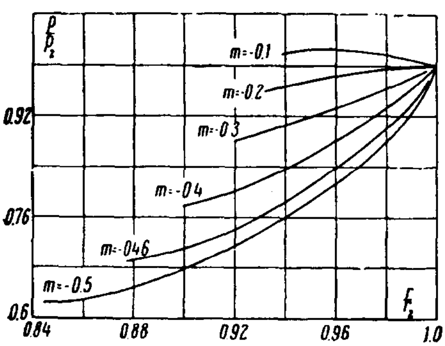

Fig. 5.

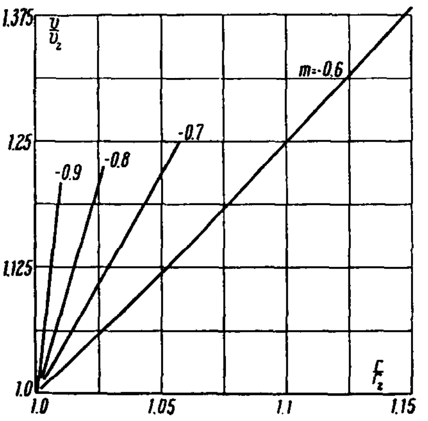

Fig. 6.

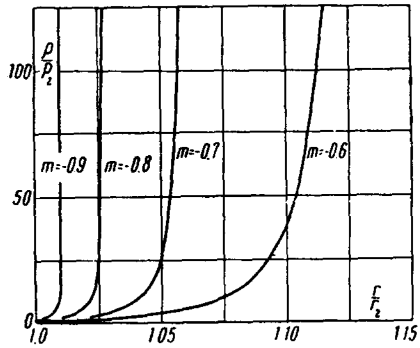

Fig. 7.

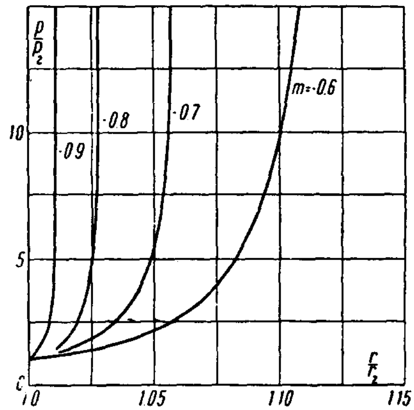

Fig. 8.

For $m = -0.1, -0.2, -0.3, -0.4, -0.46$, and -0.5 $(m' < m < 0)$ the density at the piston tends to zero, the temperature to infinity, and the pressure is finite (Fig. 3-5). For $m = -0.1$ the pressure on the piston is greater than at the shock wave; in the other cases it is less.

For $m = -0.6, -0.7, -0.8, -0.9$ $(-1 < m < m')$ a different picture of the flow is obtained. In this range of the parameter m the radius r_2 of the shock wave is smaller than the radius of the piston r_*; that is, $\lambda_2 > m + 1$. Consequently the speed of the shock wave

$$D = (m+1)/\, ct^m \lambda_2 < v_* = ct^m$$

is less than the speed of the piston.

In this case one must suppose that the mass of gas occupies the exterior of the sphere of radius

$$r_* = \frac{ct^m}{m+1}$$

expanding in the course of time. The shock wave lags behind the piston. The disturbed motion occupies the region

$$\frac{ct^m}{\lambda_2} < r < \frac{ct^m}{m+1}$$

and inside the sphere of radius

$$r_2 = \frac{ct^m}{\lambda_2}$$

the gas is at rest, the pressure is equal to zero, and the density is the initial density ρ_1. The density on the piston is infinite, the temperature is equal to zero, and the pressure is finite (Fig. 6-8).

As was mentioned in the preceding section, such a motion can be regarded as arising for $t < 0$ from a peripheral explosion with neglect of the motion of the products of explosion.

For $m = -0.6$ the calculation was made using formulas giving the ' ıct solution of the problem of a strong explosion. (We note that the n' -rical values of the characteristics of the motion in the case $m = -0.6$ .o not agree with the values obtained in Ref. 5).

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